

ON THE QALE GEOMETRY OF NAKAJIMA'S METRIC

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ABSTRACT. We show that on Hilbert scheme of n points on \mathbb{C}^2 , the hyperkähler metric constructed by H. Nakajima via hyperkähler reduction is the Quasi-Asymptotically Locally Euclidean (QALE in short) metric constructed by D. Joyce.

1. INTRODUCTION

The Hilbert scheme (or Douady scheme) of n points on \mathbb{C}^2 , denoted by $\text{Hilb}_0^n(\mathbb{C}^2)$, is a crepant resolution of the quotient of $(\mathbb{C}^2)_0^n = \{q \in (\mathbb{C}^2)^n, \sum_j q_j = 0\}$ by the action of the symmetric group S_n which acts by permutation of the indices :

$$\sigma \in S_n, q \in (\mathbb{C}^2)_0^n, \sigma.q = (q_{\sigma^{-1}(1)}, q_{\sigma^{-1}(2)}, \dots, q_{\sigma^{-1}(n)}).$$

Hence we have a map

$$\pi : \text{Hilb}_0^n(\mathbb{C}^2) \rightarrow (\mathbb{C}^2)_0^n / S_n.$$

The complex manifold $\text{Hilb}_0^n(\mathbb{C}^2)$ carries a natural complex symplectic structure which comes from the S_n invariant one of $(\mathbb{C}^2)_0^n$. A compact Kähler manifold admitting a complex symplectic form carries in his Kähler class a hyperkähler metric, this is now a well-known consequence of the solution of the Calabi conjecture by S-T. Yau (see [3]). However, $\text{Hilb}_0^n(\mathbb{C}^2)$ is non compact, for instance $\text{Hilb}_0^2(\mathbb{C}^2) = T^*\mathbb{P}^1(\mathbb{C})$. There are many extensions of Yau's result to non compact manifold (see for instance [2],[26],[27]) and in 1999, D. Joyce has introduced a new class of asymptotic geometry called Quasi-Asymptotically Locally Euclidean (QALE in short) ; this class is the extension of the class of ALE (for Asymptotically Locally Euclidean) ; roughly a complete manifold (M^d, g) is called ALE asymptotic to \mathbb{R}^d / Γ where $\Gamma \subset O(d)$ is a finite subgroup acting freely on \mathbb{S}^{d-1} , if outside a compact set M is diffeomorphic to $(\mathbb{R}^d \setminus \mathbb{B}) / \Gamma$ and if on there the metric is asymptotic to the Euclidean metric (the precise definition requires estimates between g and the Euclidean metric). When X^m is a crepant resolution of \mathbb{C}^m / Γ for $\Gamma \subset SU(m)$ a finite group, then roughly a Kähler metric on X^m is called QALE if firstly away from the pulled back of the singular set the metric is asymptotic to the Euclidean one and secondly on pieces of X^m which (up to a finite ambiguity) are diffeomorphic to a subset of $X_A \times \text{Fix}(A)$ where A is a subgroup of Γ , and X_A is a crepant resolution of $\text{Fix}(A)^\perp / A$, then the metric is asymptotic to the sum of a QALE Kähler metric on X_A and a Euclidean metric on $\text{Fix}(A)$. And D. Joyce has proved the following ([15],[16][theorem 9.3.3 and 9.3.4]):

Theorem A. *When $\Gamma \subset SU(m)$ is a finite group and $X^m \rightarrow \mathbb{C}^m / \Gamma$ is a crepant resolution , then in any Kähler class of QALE metric there is a unique QALE Kähler Ricci flat metric. Moreover if $\Gamma \subset Sp(m/2)$, then this metric is hyperkähler.*

In particular, up to scaling, $\text{Hilb}_0^n(\mathbb{C}^2)$ carries an unique hyperkähler metric asymptotic to $(\mathbb{C}^2)_0^n / S_n$.

Another fruitful construction of hyperkähler metric is the hyperkähler quotient construction of N. Hitchin, A. Karlhede, U. Lindström and M. Rocek [13]. In fact in 1999, H. Nakajima has constructed a hyperkähler metric on $\text{Hilb}_0^n(\mathbb{C}^2)$ as a hyperkähler quotient [24]. Moreover H. Nakajima asked whether this metric could be recovered via a resolution of the Calabi conjecture ; also D. Joyce said that it is likely that QALE hyperkähler metric can be explicitly constructed using the hyperkähler quotient, but outside the case of $\Gamma \subset SU(2) = Sp(1)$ treated by Kronheimer [17], he has no examples. The main result of this paper is the following :

Theorem B. *On $\text{Hilb}_0^n(\mathbb{C}^2)$, up a scaling, D.Joyce's and H.Nakajima's metrics coincide.*

It should be noted that a given complex manifold can carry two very different hyperkähler metrics ; for instance as it has been clearly explained by C. Lebrun \mathbb{C}^2 carries two quite different Kähler Ricci flat metrics, the Euclidean one and the Taub-Nut metric which has cubic volume growth [19].

The main evident idea of the proof of this result is to study the asymptotics of Nakajima's metric ; however in order to use D. Joyce's unicity result, we would need also asymptotics on the derivatives of Nakajima's metric, this is probably possible but requires more estimates. Our analysis of the asymptotics of Nakajima's metric gives that Joyce and Nakajima's metrics differ by $O(\rho^{-2}\sigma^{-2})$; where ρ is the distance to a fixed point and σ a regularized version of the distance to the singular set. And in order to use the classical argument of S-T. Yau giving the unicity of the solution to the Calabi conjecture, we need to find a function φ vanishing at infinity such that, the difference between the two Kähler forms of Nakajima and Joyce's metric is $i\partial\bar{\partial}\varphi$. D. Joyce has developed elaborate tools to solve the equation of the type $\Delta u = f$ on QALE manifold ; but the decay $O(\rho^{-2}\sigma^{-2})$ is critical for this analysis. In fact, we have circumvented this difficulty using the Li-Yau's estimates for the Green kernel of a manifold with non negative curvature [20], and we have obtained the following result, which has an independent interest and which can be generalized to other QALE manifolds :

Theorem C. *Let f is a locally bounded function on $\text{Hilb}_0^n(\mathbb{C}^2)$, such that for some $\varepsilon > 0$ satisfies*

$$f = O\left(\frac{1}{\rho^\varepsilon \sigma^2}\right)$$

then the equation $\Delta u = f$ has a unique solution such that

$$u = O\left(\frac{\log(\rho + 2)}{\rho^\varepsilon}\right).$$

For further more profound results on the analysis on QALE space, there is a very interesting work of A. Degeratu and R. Mazzeo [7].

In the physics literature, $\text{Hilb}_0^n(\mathbb{C}^2)$ is associated to the moduli space of instantons on noncommutative \mathbb{R}^4 [25]. Our motivation for the study of the asymptotic geometry of the Nakajima's metric comes from a question of C. Vafa and E. Witten about the space of L^2 harmonic forms on $\text{Hilb}_0^n(\mathbb{C}^2)$ endowed with the Nakajima's metric. Let \mathcal{H}^k be the space of L^2 harmonic k -forms on $\text{Hilb}_0^n(\mathbb{C}^2)$:

$$\mathcal{H}^k = \{\alpha \in L^2(\Lambda^k T^* \text{Hilb}_0^n(\mathbb{C}^2)), d\alpha = d^* \alpha = 0\}.$$

In [28], see also the nice survey of T. Hausel [10], the following question is asked :

Conjecture D.

$$\mathcal{H}^k = \begin{cases} \{0\} & \text{if } k \neq 2(n-1) = \dim_{\mathbb{R}} \text{Hilb}_0^n(\mathbb{C}^2) \\ \text{Im} (H_c^k(\text{Hilb}_0^n(\mathbb{C}^2)) \rightarrow H^k(\text{Hilb}_0^n(\mathbb{C}^2))) & \text{if } k = 2(n-1) \end{cases}$$

However, C. Vafa and E. Witten said "Unfortunately, we do not understand the prediction of S -duality on non-compact manifolds precisely enough to fully exploit them."

In fact, N. Hitchin has shown that the vanishing of the space of L^2 harmonics k -forms outside middle degree is a general fact for hyperkähler reduction of the flat quaternionic space \mathbb{H}^m by a compact subgroup of $Sp(m)$ [12] ; he obtained this result with a generalization of an idea of M. Gromov ([9] see also related works by J.Jost, K. Zuo and J. Mc Neal [14],[21]). For the degree $k = 2(n-1)$, the cohomology of $\text{Hilb}_0^n(\mathbb{C}^2)$ is well known :

$$\begin{aligned} H_c^{2(n-1)}(\text{Hilb}_0^n(\mathbb{C}^2)) &\simeq \text{Im} \left(H_c^{2(n-1)}(\text{Hilb}_0^n(\mathbb{C}^2)) \rightarrow H^{2(n-1)}(\text{Hilb}_0^n(\mathbb{C}^2)) \right) \\ &\simeq H^{2(n-1)}(\text{Hilb}_0^n(\mathbb{C}^2)) \simeq \mathbb{R} \end{aligned}$$

and a dual class to the generator is $\pi^{-1}\{0\}$. Moreover a general result of M. Anderson says that the image of the cohomology with compact support in the cohomology always injects inside the space of L^2 harmonics forms [1]. Hence for the Hilbert scheme of n points in \mathbb{C}^2 endowed with Nakajima's metric we always have

$$\dim \mathcal{H}^{2(n-1)} \geq 1$$

and the conjecture D predicts the equality $\dim \mathcal{H}^{2(n-1)} = 1$.

There are many results on the topological interpretation of the space of L^2 harmonic forms on non compact manifolds but all of them requires a little on the knowledge of the asymptotic geometry (see [11] for results related to some prediction from string theory and [6] for a list of such results) ; the rough idea is that this asymptotic geometry would provide a certain behavior of L^2 harmonic forms (decay, polyhomogeneity in a good compactification) and that would imply a topological interpretation of this space with a cohomology of a compactification. With our paper [5], our main result implies :

Theorem E. *The Vafa-Witten conjecture D conjecture is true when $n = 3$.*

The case $n = 2$ can be treated by explicit computation (see [12] for clever computations).

As the Vafa-Witten conjecture is in fact more general and concerns the quivers varieties constructed by H.Nakajima [23], a natural perspective is to understand the asymptotic geometry of the quivers varieties and the class of Quasi-asymptotically Conical manifolds introduced by R. Mazzeo should be usefull [22]. In a different direction it would be good to develop appropriate QALE tools to settle the status of the Vafa-Witten conjecture.

Acknowledgements. It is a pleasure to thank A. Degeratu, P. Romon, R. Mazzeo, M. Singer, C. Sorger and Y. Rollin for interesting discussion related to this work ; a special thank is due to O. Biquard who suggested that I could use the classical proof of the unicity of the solution of the Calabi conjecture in place of difficult derivative estimate. This paper was finished during a stay at the MSRI, and I was partially supported by a joint NSF-CNRS project and the project ANR project GeomEinstein 06-BLAN-0154.

2. NAKAJIMA'S METRIC

In [24], H. Nakajima has shown that the Hilbert scheme of n points in \mathbb{C}^2 carries a natural hyperkähler metric ; this metric is obtained from the Hyperkählerian quotient construction of N. Hitchin, A. Karlhede, U. Lindström and M. Rocek [13] : the complex vector space

$$\mathbb{M}_n = \mathbb{M} := \{(A, B, x, y) \in \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{M}_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^*, \text{tr } A = \text{tr } B = 0\}$$

has a complex structure

$$J(A, B, x, y) = (B^*, -A^*, y^*, -x^*)$$

if we let $K = iJ$ then $(\mathbb{M}, I = i, J, K := iJ)$ becomes a quaternionic vector space ; moreover the unitary group $U(n)$ acts linearly on \mathbb{M} : if $g \in U(n)$ and $z = (A, B, x, y) \in \mathbb{M}$ then

$$g.z = (gAg^{-1}, gBg^{-1}, gx, gy^{-1}).$$

The real moment map associated to this action is

$$\mu(A, B, x, y) = \frac{1}{2i} ([A, A^*] + [B, B^*] + xx^* - yy^*) \in \mathfrak{u}(n).$$

If $h \in \mathfrak{u}(n)$ and $z = (A, B, x, y) \in \mathbb{M}$ we let

$$l_z(h) = \left. \frac{d}{dt} \right|_{t=0} e^{th}.z = ([h, A], [h, B], hx, -yh).$$

By definition, we have for $z \in \mathbb{M}, \delta z \in T_z \mathbb{M} \simeq \mathbb{M}$:

$$\langle d\mu(z)(\delta z), h \rangle = \langle il_z(h), \delta z \rangle$$

The action of $GL_n(\mathbb{C})$ on \mathbb{M} preserves the complex symplectic form

$$\omega_{\mathbb{C}}(z, z') = \text{tr}(A.B' - B.A') + y'(x) - y(x'),$$

and the associated complex moment map is :

$$\mu_{\mathbb{C}}(A, B, x, y) = [A, B] + xy \in \mathcal{M}_n(\mathbb{C}).$$

Let $t > 0$ and defined

$$\mathbb{L}_t(n) = \mathbb{L}_t := \mu^{-1} \left\{ \frac{t}{2i} \right\} \cap \mu_{\mathbb{C}}^{-1}\{0\},$$

then the map

$$\#\mu := (\mu, \mu_{\mathbb{C}}) : \mathbb{M} \rightarrow \mathfrak{u}(n) \oplus \mathcal{M}_n(\mathbb{C})$$

is a submersion near \mathbb{L}_t and $U(n)$ acts freely on it, hence the quotient $\mathbb{H}_t := \mathbb{L}_t/U(n)$ is a smooth manifold, this manifold is endowed with the Riemannian metric g_N which makes the submersion $\mathbb{L}_t \rightarrow \mathbb{H}_t$ Riemannian. By definition the tangent space of $U(n)z$ is naturally isometric to the orthogonal of the space

$$\text{Im } l_z \oplus I \text{Im } l_z \oplus J \text{Im } l_z \oplus K \text{Im } l_z.$$

In particular, \mathbb{H}_t is endowed with a quaternionic structure which is in fact integrable ; hence the metric g_N is hyperkähler hence Kähler and Ricci flat.

2.1. Some remarks. Because for $\lambda > 0$, we have $\lambda \mathbb{L}_t = \mathbb{L}_{\lambda^2 t}$, all the spaces $\{\mathbb{H}_t\}_{t>0}$ are isomorphic and their Riemannian metrics are proportional.

For $t = 0$, the quotient $\mathbb{L}_0/U(n)$ is not a smooth manifold. It is easy to show that

$$(A, B, x, y) \in \mathbb{L}_0 \Leftrightarrow (x = 0, y = 0, [A, B] = [A, A^*] = [B, B^*] = 0);$$

hence to $(A, B, 0, 0) \in \mathbb{L}_0$ we can associate their joint spectrum

$$S_n.((\lambda_1, \mu_1), (\lambda_2, \mu_2), \dots, (\lambda_n, \mu_n)) \in (\mathbb{C}^2)_0^n / S_n$$

where $(\mathbb{C}^2)_0^n := \{(q_1, \dots, q_n \in (\mathbb{C}^2)^n, \sum_j q_j = 0\}$ and the symmetric group S_n acts on $(\mathbb{C}^2)_0^n$ by permutation of the indices. We get an isomorphism (in fact an isometry)

$$\mathbb{L}_0/U(n) \simeq (\mathbb{C}^2)_0^n / S_n.$$

In fact for $t > 0$, we still have

$$(2.1) \quad (A, B, x, y) \in \mathbb{L}_t \Rightarrow y = 0.$$

Hence for $z := (A, B, x, 0) \in \mathbb{L}_t$, the joint spectrum of (A, B) is still defined and we can defined

$$\pi(U(n)z) = S_n.((\lambda_1, \mu_1), (\lambda_2, \mu_2), \dots, (\lambda_n, \mu_n)) \in (\mathbb{C}^2)_0^n / S_n$$

where $(\lambda_1, \mu_1), (\lambda_2, \mu_2), \dots, (\lambda_n, \mu_n)$ are such that for a $g \in U(n)$, the matrix gAg^{-1} (resp. gBg^{-1}) is upper triangular with diagonal $(\lambda_1, \lambda_2, \dots, \lambda_n)$ (resp. $(\mu_1, \mu_2, \dots, \mu_n)$). \mathbb{H}_t is isomorphic to the Hilbert scheme of n points¹ in \mathbb{C}^2 $\text{Hilb}_0^n(\mathbb{C}^2)$. The map $\pi : \text{Hilb}_0^n(\mathbb{C}^2) \rightarrow (\mathbb{C}^2)_0^n / S_n$ is in fact a crepant resolution of $(\mathbb{C}^2)_0^n / S_n$.

Remark 2.1. We also remark that if $v = (\delta A, \delta B, \delta x, 0) \in T_\zeta \mathbb{L}_t$ is orthogonal to the range of l_ζ , then Jv is also in $T_\zeta \mathbb{L}_t$, and hence $\delta x = 0$.

2.2. The geometry of $\text{Hilb}_0^2(\mathbb{C}^2)$. As an example, we look at the geometry of $\text{Hilb}_0^2(\mathbb{C}^2)$. Let $z = (A, B, x, 0) \in \mathcal{M}_2(\mathbb{C}) \oplus \mathcal{M}_2(\mathbb{C}) \oplus \mathbb{C}^2 \oplus (\mathbb{C}^2)^*$ such that $\text{tr } A = \text{tr } B = 0$ and

$$\begin{cases} [A, A^*] + [B, B^*] + xx^* = t \text{ Id} \\ [A, B] = 0 \end{cases}.$$

When $\det A \neq 0$ or $\det B \neq 0$ then we can find a $g \in U(2)$ such that

$$gAg^{-1} = \begin{pmatrix} \lambda & a \\ 0 & -\lambda \end{pmatrix}, gBg^{-1} = \begin{pmatrix} \mu & b \\ 0 & -\mu \end{pmatrix}$$

then let $g(x) = (x_1, x_2)$. The equation $[A, B] = 0$ implies that there is a number ρ such that $a = \lambda\rho$ and $b = \mu\rho$. Then the remaining equations are for $R^2 := |\lambda|^2 + |\mu|^2$

$$\begin{cases} |\rho|^2 R^2 + |x_1|^2 = t \\ -|\rho|^2 R^2 + |x_2|^2 = t \\ -2R^2\rho + x_1\overline{x_2} = 0 \end{cases}$$

We can always choose g such that $x_1, x_2 \in \mathbb{R}_+$ then we obtain $\rho^2 = \sqrt{4 + \frac{t^2}{R^4}} - 2$ Hence

$$\rho = \frac{t}{2R^2} + O\left(\frac{1}{R^6}\right)$$

¹with the center of mass removed.

if $x_1 = \sqrt{2t} \sin(\phi)$, $x_2 = \sqrt{2t} \cos(\phi)$, then $t \cos(2\phi) = \rho^2 R^2$ and $t \sin(2\phi) = 2\rho R^2$. Hence $\phi = \frac{\pi}{4} + O\left(\frac{1}{R}\right)$ and $x_1 = \sqrt{\frac{t}{2}} + O\left(\frac{1}{R}\right)$, $x_2 = \sqrt{\frac{t}{2}} + O\left(\frac{1}{R}\right)$. Hence for $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\} \simeq (\mathbb{C}^2)_0^2$ we have found

$$z(\lambda, \mu) = \left(\begin{pmatrix} \lambda & \lambda\rho(R) \\ 0 & -\lambda \end{pmatrix}, \begin{pmatrix} \mu & \mu\rho(R) \\ 0 & -\mu \end{pmatrix}, x(R), 0 \right) \in \mathbb{L}_t.$$

Moreover, $z(\lambda, \mu)$ and $z(\lambda', \mu')$ are in the same $U(2)$ orbit if and only if $(\lambda, \mu) = \pm(\lambda', \mu')$; hence we have a map

$$z : (\mathbb{C}^2 \setminus \{0\}) / \{\pm Id\} \rightarrow \mathbb{L}_t / U(2).$$

From the exact value of z , we can show that

$$z^* g_N = 2 \left[|d\lambda|^2 + |d\mu|^2 \right] + O\left(\frac{1}{R^4}\right).$$

This shows that $(\text{Hilb}_0^2(\mathbb{C}^2), g_N)$ is a hyperkähler metric which is Asymptotically Locally Euclidean asymptotic to $\mathbb{C}^2 / \{\pm Id\}$. These manifolds has been classified by Kronheimer [18], so that in this case Nakajima's metric is the Eguchi-Hansen metric on $T^*\mathbb{P}^1(\mathbb{C})$.

2.3. A last useful remark. A priori, it is not clear wether the above map z is holomorphic, this is in fact true as a consequence of the following useful lemma :

Lemma 2.2. *Suppose that a compact Lie group G acts on \mathbb{H}^m by quaternionic linear maps and let $\mu : \mathbb{H}^m \rightarrow \mathfrak{g}^* \otimes \text{Im } \mathbb{H}$ be the associated moment map. Assume that for some $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in \mathfrak{g}^* \otimes \text{Im } \mathbb{H}$ the hyperkähler quotient $Q := \mu^{-1}\{\zeta\}/G$ is well defined. When X is a complex manifold and $\Psi : X \rightarrow \mu^{-1}\{\zeta\}$ is a smooth map such that locally*

$$\Psi(x) = g(x)\tilde{\Psi}(x)$$

where $g : X \rightarrow G^{\mathbb{C}}$ is smooth and $\tilde{\Psi} : X \rightarrow \mu_{\mathbb{C}}^{-1}\{\zeta_{\mathbb{C}}\}$ is holomorphic, then the induced map $\bar{\Psi} : X \rightarrow Q$ is also holomorphic.

Proof. If $q \in \mathbb{H}^m$ let P_q be the orthogonal projection onto the orthogonal of $\text{Im } l_q \oplus I \text{Im } l_q \oplus J \text{Im } l_q \oplus K \text{Im } l_q = \text{Im } l_q^{\mathbb{C}} \oplus J \text{Im } l_q^{\mathbb{C}}$ where $l_q : \mathfrak{k} \rightarrow \mathbb{H}^q$ is defined as before by

$$l_q(h) = \frac{d}{dt} \Big|_{t=0} e^{th} \cdot q = h \cdot q.$$

We must show that if $x \in X$, then for $q := \Psi(x)$:

$$P_q(d\Psi(x)(Iv)) = IP_q(d\Psi(x)(v)).$$

But $\dot{g}(x) = dg(x)(Iv) \in G^{\mathbb{C}}$ we have

$$d\Psi(x)(Iv) = \dot{g}(x) \cdot q + g(x) \cdot d\tilde{\Psi}(x)(Iv) = l_q^{\mathbb{C}}(\dot{g}(x)) + g(x) \cdot d\tilde{\Psi}(x)(Iv)$$

By definition $P_q(l_q^{\mathbb{C}}(\dot{g}(x))) = 0$ and because $g(x)$ and P_q are complex linear :

$$P_q(d\Psi(x)(Iv)) = P_q(g(x) \cdot Id\tilde{\Psi}(x)(v)) = IP_q(d\Psi(x)(v)).$$

□

3. JOYCE'S METRIC

In [15, 16], D. Joyce has build many new Kähler, Ricci flat metrics on some crepant resolution of quotient of \mathbb{C}^m by a finite subgroup of $SU(m)$; his construction relies upon the resolution of a Calabi-Yau problem for a certain class of asymptotic geometry which is called QALE for Quasi Asymptotically Locally Euclidean. We will follow the presentation of D. Joyce for the Hilbert Scheme of n points on \mathbb{C}^2 and we will then describe the asymptotic geometry of these QALE metrics on $\text{Hilb}_0^n(\mathbb{C}^2)$.

3.1. The local product resolution of $\text{Hilb}_0^n(\mathbb{C}^2)$. If $\mathfrak{p} = (I_1, I_2, \dots, I_k)$ is a partition of $\{1, 2, \dots, n\}$ ², the I_l 's are called the cluster of \mathfrak{p} . We will denote

$$V_{\mathfrak{p}} = \{q \in (\mathbb{C}^2)_0^n, \forall l \in \{1, \dots, k\}, \forall i, j \in I_l : q_i = q_j\}$$

and $A_{\mathfrak{p}} = \{\gamma \in S_n, \gamma q = q \forall q \in V_{\mathfrak{p}}\} \simeq S_{n_1} \times S_{n_2} \times \dots \times S_{n_k}$ where $n_l = \#I_l$. Then

$$W_{\mathfrak{p}} = V_{\mathfrak{p}}^{\perp} \simeq \bigoplus_{l=1}^k (\mathbb{C}^2)_0^{n_l}.$$

Let $m_{\mathfrak{p}} = \text{codim}_{\mathbb{C}} V_{\mathfrak{p}} = \dim_{\mathbb{C}} W_{\mathfrak{p}} = 2(n - l(\mathfrak{p}))$ where $l(\mathfrak{p}) = k$. The set \mathcal{P}_n of partitions of $\{1, 2, \dots, n\}$ has the following partial order :

$$\mathfrak{p} \leq \mathfrak{q} \Leftrightarrow V_{\mathfrak{q}} \subset V_{\mathfrak{p}} \Leftrightarrow W_{\mathfrak{p}} \subset W_{\mathfrak{q}}$$

Hence $\mathfrak{p} \leq \mathfrak{q}$ if and only if \mathfrak{p} is a refinement of \mathfrak{q} : i.e. if $\mathfrak{q} = (J_1, J_2, \dots, J_r)$, then there are partitions $(I_{l,1}, I_{l,2}, \dots, I_{l,n_l})$ of $J_l = I_{l,1} \cup \dots \cup I_{l,n_l}$ such that the cluster of \mathfrak{p} are the $I_{l,j}$'s. The smallest partition is $\mathfrak{p}_0 = \{1\} \cup \{2\} \cup \dots \cup \{n\}$ with $V_{\mathfrak{p}_0} = (\mathbb{C}^2)_0^n$, the largest partition is $\mathfrak{p}_{\infty} = \{1, 2, \dots, n\}$ with $V_{\mathfrak{p}_{\infty}} = \{0\}$. The fundamental partitions are the $\mathfrak{p}_{i,j}$ such that

$$\mathfrak{p}_{i,j} = (\{i, j\}, \{k_1\}, \{k_2\}, \dots, \{k_{n-2}\})$$

with $\{1, 2, \dots, n\} \setminus \{i, j\} = \{k_1, k_2, \dots, k_{n-2}\}$, then $V_{i,j} := V_{\mathfrak{p}_{i,j}} = \{q \in (\mathbb{C}^2)_0^n, q_i = q_j\}$. We have for any partition $\mathfrak{p} \neq \mathfrak{p}_0$

$$V_{\mathfrak{p}} = \cap_{\mathfrak{p}_{i,j} \leq \mathfrak{p}} V_{i,j}$$

We will also denote $\Delta_{\mathfrak{p}} = \{(i, j) \in \{1, 2, \dots, n\}^2, \mathfrak{p}_{i,j} \not\leq \mathfrak{p}\}$ and $\Delta_{\mathfrak{p}}^c = \{(i, j) \in \{1, 2, \dots, n\}^2, \mathfrak{p}_{i,j} \leq \mathfrak{p}\}$. The singular locus of $(\mathbb{C}^2)_0^n / S_n$ is the quotient of the generalized diagonal

$$S = \left(\bigcup_{\mathfrak{p} \neq \mathfrak{p}_0} V_{\mathfrak{p}} \right) / S_n = \left(\bigcup_{i,j} V_{i,j} \right) / S_n.$$

Finally let

$$S_{\mathfrak{p}} = \left(\bigcup_{(i,j) \in \Delta_{\mathfrak{p}}} V_{i,j} \right) / A_{\mathfrak{p}}$$

and for $R > 0$, let $T_{\mathfrak{p}}$ be the R -neighborhood of $S_{\mathfrak{p}}$ is $(\mathbb{C}^2)_0^n / A_{\mathfrak{p}}$:

$$T_{\mathfrak{p}} := \{q \in (\mathbb{C}^2)_0^n, \exists (i, j) \in \Delta_{\mathfrak{p}} |q_i - q_j| < R\} / A_{\mathfrak{p}}.$$

²the I_l 's are disjoint and their reunion is $\{1, 2, \dots, n\}$.

The resolution $\pi : \text{Hilb}_0^n(\mathbb{C}^2) \rightarrow (\mathbb{C}^2)_0^n / S_n$ is a local product resolution ; indeed there is a resolution of $W_{\mathfrak{p}}/A_{\mathfrak{p}}$ namely

$$\pi_{\mathfrak{p}} : \text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) := \prod_{l=1}^k \text{Hilb}_0^{n_l}(\mathbb{C}^2) \rightarrow W_{\mathfrak{p}}/A_{\mathfrak{p}}$$

such that for $U_{\mathfrak{p}} = (\pi_{\mathfrak{p}} \times \text{Id})^{-1}(T_{\mathfrak{p}}) \subset \text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) \times V_{\mathfrak{p}}$ and $\phi_{\mathfrak{p}} : (\mathbb{C}^2)_0^n / A_{\mathfrak{p}} \rightarrow (\mathbb{C}^2)_0^n / S_n$ the natural map there is a local biholomorphism onto his image $\psi_{\mathfrak{p}} : U_{\mathfrak{p}} \rightarrow \text{Hilb}_0^n(\mathbb{C}^2)$ for which the following diagram is commutative :

$$\begin{array}{ccc} \text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) \times V_{\mathfrak{p}} \setminus U_{\mathfrak{p}} & \xrightarrow{\psi_{\mathfrak{p}}} & \text{Hilb}_0^n(\mathbb{C}^2) \\ \downarrow \pi_{\mathfrak{p}} \times \text{Id} & & \downarrow \pi \\ (\mathbb{C}^2)_0^n / A_{\mathfrak{p}} \setminus T_{\mathfrak{p}} & \xrightarrow{\phi_{\mathfrak{p}}} & (\mathbb{C}^2)_0^n / S_n \end{array}$$

In the hyperkählerian quotient description, the local biholomorphism $\psi_{\mathfrak{p}}$ is given as follows identifying $V_{\mathfrak{p}}$ with $(\mathbb{C}^2)_0^k$, if we let

$$\zeta = ((A_1, B_1, x_1, 0), (A_2, B_2, x_2, 0), \dots, (A_k, B_k, x_k, 0)) \in \prod_{j=1}^k \mathbb{L}_t(n_j)$$

and $\eta = ((\lambda_1, \mu_1), (\lambda_2, \mu_2), \dots, (\lambda_k, \mu_k)) \in (\mathbb{C}^2)_0^k \setminus U_{\mathfrak{p}}$, we associated to (ζ, η) , the vector $(A, B, x, 0) \in \mathbb{M}(n)$ such that A and B are block diagonal with respective diagonal $(A_1 + \lambda_1, A_2 + \lambda_2, \dots, A_k + \lambda_k)$ and $(B_1 + \mu_1, B_2 + \mu_2, \dots, B_k + \mu_k)$ and $x = (x_1, x_2, \dots, x_k)$ then $\psi_{\mathfrak{p}}((U(n_1) \times U(n_2) \times \dots \times U(n_k)) \cdot \zeta, \eta)$ is the set of points, in the $GL_n(\mathbb{C})$ -orbit of $(A, B, x, 0)$, satisfying the real moment map equation (see the part 4 for more details).

3.2. QALE metric on $\text{Hilb}_0^n(\mathbb{C}^2)$. We introduce several functions of distance's type on $(\text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) \times V_{\mathfrak{p}}) \setminus U_{\mathfrak{p}}$, if $z \in \text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) \times V_{\mathfrak{p}} \setminus U_{\mathfrak{p}}$ and $v = (\pi_{\mathfrak{p}} \times \text{Id})(z)$, we note

$$\mu_{\mathfrak{p}, \mathfrak{q}}(z) = \inf_{\gamma \in A_{\mathfrak{p}}} d(\gamma \cdot v, V_{\mathfrak{q}}) = d(v, (A_{\mathfrak{p}} V_{\mathfrak{q}}) / A_{\mathfrak{p}})$$

and

$$\nu_{\mathfrak{p}}(z) = 1 + \inf_{p \neq p_0} \mu_{\mathfrak{p}, \mathfrak{q}}(z)$$

Then a Riemannian metric g on $\text{Hilb}_0^n(\mathbb{C}^2)$ is called QALE (asymptotic to $(\mathbb{C}^2)_0^n / S_n$) if for each partition \mathfrak{p} there is a metric $g_{\mathfrak{p}}$ on $\text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)$ such that for all $l \in \mathbb{N}$:

$$(3.1) \quad \nabla^l (\psi_{\mathfrak{p}}^* g - (g_{\mathfrak{p}} + \text{eucl}_{V_{\mathfrak{p}}})) = \sum_{\mathfrak{q} \not\leq \mathfrak{p}} O \left(\frac{1}{\nu_{\mathfrak{p}}^{2+l} \mu_{\mathfrak{p}, \mathfrak{q}}^{2m_{\mathfrak{q}}-2}} \right)$$

However, if $\mathfrak{q} \not\leq \mathfrak{p}$ there is always a $(i, j) \in \Delta_{\mathfrak{p}}$ such that $\mathfrak{p}_{i,j} \not\leq \mathfrak{p}$ and $\mathfrak{p}_{i,j} \leq \mathfrak{q}$, therefore

$$\mu_{\mathfrak{p}, \mathfrak{p}_{i,j}}^{2m_{\mathfrak{p}_{i,j}}-2} = \mu_{\mathfrak{p}, \mathfrak{p}_{i,j}}^2 \leq \mu_{\mathfrak{p}, \mathfrak{q}}^{2m_{\mathfrak{q}}-2}$$

If we introduce $\rho_{\mathfrak{p}}(z) = \inf_{(i,j) \in \Delta_{\mathfrak{p}}} \mu_{\mathfrak{p}, \mathfrak{p}_{i,j}}$ then in fact for $v = (\pi_{\mathfrak{p}} \times \text{Id})(z) \in (\mathbb{C}^2)_0^n / A_{\mathfrak{p}}$ we have

$$\rho_{\mathfrak{p}}(z) = \inf_{(i,j) \in \Delta_{\mathfrak{p}}} |v_i - v_j|$$

The asymptotic 3.1 are equivalent to

$$(3.2) \quad \nabla^l (\psi_{\mathfrak{p}}^* g - (g_{\mathfrak{p}} + \text{eucl}_{V_{\mathfrak{p}}})) = O\left(\frac{1}{\nu_{\mathfrak{p}}^{2+l} \rho_{\mathfrak{p}}^2}\right)$$

We can introduce two functions of distance's type : when $z \in \text{Hilb}_0^n(\mathbb{C}^2)$ and $\pi(z) = (v_1, v_2, \dots, v_n) \in (\mathbb{C}^2)_0^n / S_n$, then we let

$$\rho(z) = \sqrt{\sum_{i < j} |v_i - v_j|^2},$$

and

$$\sigma(z) = \inf_{i \neq j} \{|v_i - v_j|\} + 1$$

If \mathfrak{p} is a partition of $\{1, 2, \dots, n\}$, and ϵ, τ, R are positive real numbers, then we introduce

$$\check{\mathcal{C}}_{\mathfrak{p}_0} = \{(v_1, \dots, v_n) \in (\mathbb{C}^2)_0^n / S_n, \text{ such that } |v| > R \text{ and } \forall i \neq j, |v_i - v_j| > \epsilon |v|\}$$

$$\check{\mathcal{C}}_{\mathfrak{p}} = \{(v_1, \dots, v_n) \in (\mathbb{C}^2)_0^n / S_n, \text{ such that } |v| > R, \forall (i, j) \in \Delta_{\mathfrak{p}} |v_i - v_j| > \sqrt{\frac{2}{n(n-1)}} |v| \text{ and } \forall (i, j) \in \Delta_{\mathfrak{p}}^c, |v_i - v_j| < 2\epsilon |v|\}$$

It is clear that if ϵ is small enough then the

$$(\mathbb{C}^2 \setminus R\mathbb{B})_0^n / S_n = \cup_{\mathfrak{p}} \phi_{\mathfrak{p}}(\check{\mathcal{C}}_{\mathfrak{p}}).$$

Moreover on $\mathcal{C}_{\mathfrak{p}} := (\pi_{\mathfrak{p}} \times \text{Id})^{-1}(\check{\mathcal{C}}_{\mathfrak{p}})$, the asymptotic 3.2 are

$$(3.3) \quad \nabla^l (\psi_{\mathfrak{p}}^* g - (g_{\mathfrak{p}} + \text{eucl}_{V_{\mathfrak{p}}})) = O\left(\frac{1}{\sigma^{2+l} \rho^2}\right)$$

Remark 3.1. It can be shown that if all metric $g_{\mathfrak{p}}$ are QALE and if the estimate 3.3 is satisfied then g is also QALE.

3.3. Joyce's result. The result of D. Joyce concerning the Hilbert scheme of n points on \mathbb{C}^2 is the following :

Theorem 3.2. *Up to scaling, $\text{Hilb}_0^n(\mathbb{C}^2)$ has a unique hyperkähler metric which is QALE asymptotic to $(\mathbb{C}^2)_0^n / S_n$.*

4. ASYMPTOTIC OF NAKAJIMA'S METRIC

4.1. Induction's hypothesis. In this part, we will prove the following result by induction on n :

- i) On $\text{Hilb}_0^n(\mathbb{C}^2)$, Nakajima's metric g_N satisfies the estimate (3.3) for $l = 0$, more precisely if $g_{\mathfrak{p}}$ is the sum of Nakajima's metric on $\text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)$, then for all partition \mathfrak{p} then for $\epsilon > 0$ small enough and R large enough, we have on $(\pi_{\mathfrak{p}} \times \text{Id})^{-1}(\check{\mathcal{C}}_{\mathfrak{p}})$

$$\psi_{\mathfrak{p}}^*(g_N) - g_{\mathfrak{p}} + \text{eucl}_{V_{\mathfrak{p}}} = O\left(\frac{1}{\sigma^2 \rho^2}\right)$$

- ii) There is a constant C such that if $z = (A, B, x, 0) \in \mathbb{L}_t$ then

$$\forall h \in \mathfrak{u}_n, \|l_z(h)\|^2 = \|h, A\|^2 + \|h, B\|^2 + \|hx\|^2 \geq C\|h\|^2$$

iii) There is a constant M such that for all $z \in \mathbb{L}_t$ and $(\delta A, \delta B, 0, 0) \in T_z \mathbb{L}_t$ orthogonal to $\text{Im } l_z$ then

$$\|[\delta A, \delta A^*]\| + \|[\delta B, \delta B^*]\| \leq \frac{M}{\sigma^2} (\|\delta A\|^2 + \|\delta B\|^2).$$

It is easy to check these three conditions for $\text{Hilb}_0^2(\mathbb{C}^2)$ thanks to the explicit description of \mathbb{L}_t in this case. So we now assume that these induction hypothesis are true for all $m < n$.

4.2. The case of well separated points. We first examine the easiest case corresponding to \mathfrak{p}_0 . More precisely, we consider $\mathbf{q} = (q_1, q_2, \dots, q_n) \in (\mathbb{C}^2)_0^n$ such that for all $i \neq j$, then $|q_i - q_j| > R$ (R will be chosen large enough), the set of such q 's will be denote by \mathcal{O}_0 . If $q_j = (\lambda_j, \mu_j)$ we search a solution $z = (A, B, x, 0) \in \mathbb{M}$ of the equation

$$(4.1) \quad \begin{cases} [A, A^*] + [B, B^*] + xx^* = t \text{ Id} \\ [A, B] = 0 \end{cases}$$

Where A, B are upper triangular matrices with respective diagonals $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $(\mu_1, \mu_2, \dots, \mu_n)$ and upper diagonal coefficients $\mathbf{a} = (a_{i,j}), \mathbf{b} = (b_{i,j})$. We obtain the following equation³ for the (i, j) coefficients of the equation (4.1) :

$$(4.2) \quad \begin{cases} (\bar{\lambda}_i - \bar{\lambda}_j)a_{i,j} + (\bar{\mu}_i - \bar{\mu}_j)b_{i,j} + \sum_k [\bar{a}_{k,i}a_{k,j} + \bar{b}_{k,i}b_{k,j} - a_{i,k}\bar{a}_{j,k} - b_{i,k}\bar{b}_{j,k}] = x_i\bar{x}_j \\ -(\bar{\mu}_i - \bar{\mu}_j)a_{i,j} + (\lambda_i - \lambda_j)b_{i,j} + \sum_k [a_{i,k}b_{k,j} - b_{i,k}a_{k,j}] = 0 \end{cases}$$

And the equation for the diagonal coefficient (i, i) of (4.1) gives :

$$(4.3) \quad \sum_k [|a_{i,k}|^2 - |a_{k,i}|^2 + |b_{i,k}|^2 - |b_{k,i}|^2] + |x_i|^2 = t$$

We let $R_{i,j} = \sqrt{|\lambda_i - \lambda_j|^2 + |\mu_i - \mu_j|^2}$ and

$$(4.4) \quad \begin{cases} x_i^0 = \sqrt{t} \\ a_{i,j}^0 = (\lambda_i - \lambda_j) \frac{t}{R_{i,j}^2} \\ b_{i,j}^0 = (\mu_i - \mu_j) \frac{t}{R_{i,j}^2} \end{cases}$$

Then if we write the equations (4.2,4.3) in the synthetic form

$$F(\mathbf{q}, \mathbf{a}, \mathbf{b}, \mathbf{x}) = 0$$

where $F : (\mathbb{C}^2)_0^n \times \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^n \rightarrow \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^n$. We have

$$F(\mathbf{q}, \mathbf{a}^0, \mathbf{b}^0, \mathbf{x}^0) = O(\sigma^{-2})$$

Moreover it is easy to check that when σ is large enough, the partial derivative in the last three argument $D_{(\mathbf{a}, \mathbf{b}, \mathbf{x})} F(\mathbf{q}, \mathbf{a}^0, \mathbf{b}^0, \mathbf{x}^0)$ is invertible and the norm of the inverse is uniformly bounded. The map F being polynomial of degré 2 in its arguments, the implicit function theorem implies that the equations (4.2,4.3) have a unique solution such that

$$(4.5) \quad (\mathbf{a}, \mathbf{b}, \mathbf{x})(q) = (\mathbf{a}^0, \mathbf{b}^0, \mathbf{x}^0) + O(\sigma^{-2}).$$

Moreover $D_q(\mathbf{a}, \mathbf{b}, \mathbf{x})(q) = O(\sigma^{-2})$. We have then build a map

$$\begin{aligned} \widehat{\Psi}_0 : \mathcal{O}_0 &\rightarrow \mathbb{L}_t \\ q &\mapsto (A(\mathbf{q}), B(\mathbf{q}), x(\mathbf{q}), 0) \end{aligned}$$

³with the convention that $a_{i,j} = b_{i,j} = 0$ if $j \leq i$.

Moreover $\widehat{\Psi}_0(\mathbf{q})$ and $\widehat{\Psi}_0(\mathbf{q}')$ live in the same $U(n)$ -orbit if and only if \mathbf{q} and \mathbf{q}' live in the same S_n -orbit hence $\widehat{\Psi}_0$ induces a map

$$\Psi_0 : \mathcal{O}_0/S_n \rightarrow \text{Hilb}_0^n(\mathbb{C}^2)$$

which is holomorphic according to the lemma (2.2). With (4.5), we have

$$(4.6) \quad |d\Psi_0(\mathbf{q}).v|^2 = |v|^2 + O\left(\frac{|v|^2}{\sigma^4}\right)$$

The first term comes from the diagonals of A and B the second one from the off-diagonal terms and the derivative of \mathbf{x} . In order to check the point i) of the induction hypothesis we must show that

$$|d\Psi_0(\mathbf{q}).v|^2 - |\Pi_z(d\Psi_0(\mathbf{q}).v)|^2 = |v|^2 + O\left(\frac{|v|^2}{\sigma^4}\right).$$

Where if $\widehat{\Psi}_0(q) = z \in \mathbb{L}_t$, Π_z is the orthogonal projection onto the space $\text{Im } l_z$. But by construction, if $X \in \text{Im } l_z$ then IX is normal to $T_z \mathbb{L}_t$ hence $d\Psi_0(\mathbf{q}).(Ix) \perp IX$ in particular $\Pi_z(I.d\Psi_0(\mathbf{q}).(Ix)) = 0$; Hence

$$(4.7) \quad \Pi_z(d\Psi_0(\mathbf{q}).v) = \Pi_z(d\Psi_0(\mathbf{q}).v + Id\Psi_0(\mathbf{q}).Ix) = 2\Pi_z(\bar{\partial}\Psi_0(\mathbf{q}).v).$$

But by construction

$$(4.8) \quad |\bar{\partial}\Psi_0(\mathbf{q}).v|^2 = |\bar{\partial}\mathbf{a}|^2 + |\bar{\partial}\mathbf{b}|^2 + |\bar{\partial}\mathbf{x}|^2 = O\left(\frac{|v|^2}{\sigma^4}\right)$$

The assertion i) of the induction hypothesis i) follows from the estimates (4.6,4.7,4.8).

For the induction hypothesis ii), we have for $z = \widehat{\Psi}_0(\mathbf{q})$ and $h = (h_{i,j}) \in \mathfrak{u}_n$:

$$\|l_z(h)\|^2 \geq \frac{1}{2} \left[\sum_{i,j} R_{i,j}^2 |h_{i,j}|^2 + t \sum_i |h_{i,i}|^2 \right] - C\sigma^{-2} \|h\|^2$$

Hence if R is chosen large enough the induction hypothesis ii) hold on \mathcal{O}_0 .

Now we check the induction hypothesis iii) let

$$(\delta A, \delta B, 0, 0) = d\Psi_0(\mathbf{q}).v - \Pi_z(d\Psi_0(\mathbf{q}).v)$$

we have just said that

$$\|\Pi_z(d\Psi_0(\mathbf{q}).v)\| = O(\sigma^{-2}) |v|.$$

Hence the off-diagonal part of δA and δB are bounded by $O(\sigma^{-2}) |v|$, this implies that

$$\|[\delta A, \delta A^*]\|^2 + \|[\delta B, \delta B^*]\|^2 \leq O(\sigma^{-4}) |v|^4.$$

4.3. The general case. We examine now the region \mathcal{C}_p associated to another partition $p \neq p_0$, we can always assume that

$$p = (\{m_0 = 1, \dots, m_1\}, \{m_1 + 1, m_2\}, \dots, \{m_{k-1} + 1, \dots, n = m_k\}),$$

let $n_l = m_l - m_{l-1}$. We consider the set \mathcal{O}_p of

$$(\mathbf{q}, \mathbf{A}, \mathbf{B}, \mathbf{x}) \in (\mathbb{C}^2)_0^k \times \bigoplus_{j=1}^k \mathcal{M}_{n_j}(\mathbb{C}) \times \bigoplus_{j=1}^k \mathcal{M}_{n_j}(\mathbb{C}) \times \bigoplus_{j=1}^k \mathbb{C}^{n_j}$$

such that if $\mathbf{q} = (q_1, q_2, \dots, q_k)$ then for all $i \neq j$ then $|q_i - q_j| > \sqrt{\frac{1}{n(n-1)}}|\mathbf{q}|$ and $|\mathbf{q}| \geq R$ and if $\mathbf{A} = (A_1, A_2, \dots, A_k)$, $\mathbf{B} = (B_1, B_2, \dots, B_k)$, $\mathbf{x} = (x_1, x_2, \dots, x_k)$ then each (A_j, B_j, x_j) satisfies $\text{tr } A_j = \text{tr } B_j = 0$ and the moment map equation :

$$\begin{cases} [A_j, A_j^*] + [B_j, B_j^*] + x_j x_j^* = t \text{Id}_{n_j} \\ [A_j, B_j] = 0 \end{cases}$$

and moreover

$$\sup_j \|A_j\|^2 + \|B_j\|^2 \leq \tau^2 |\mathbf{q}|^2$$

We will search a solution $z = (A, B, x, 0)$ of the moment map equation which is approximatively

$$A \simeq \begin{pmatrix} A_1 + \lambda_1 & 0 & \dots & 0 \\ 0 & A_2 + \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & A_k + \lambda_k \end{pmatrix},$$

$$B \simeq \begin{pmatrix} B_1 + \mu_1 & 0 & \dots & 0 \\ 0 & B_2 + \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & B_k + \mu_k \end{pmatrix},$$

$$x \simeq (x_1, x_2, \dots, x_k)$$

We first fix some $\zeta = (\mathbf{q}, \mathbf{A}, \mathbf{B}, \mathbf{x}) \in \mathcal{O}_{\mathbf{p}}$ and we search a $z_0 = (A^0, B^0, x^0, 0)$ where if $q_j = (\lambda_j, \mu_j)$ then $x^0 = (x_1, x_2, \dots, x_k)$, A^0 (resp. B^0) is upper block triangular with diagonal $(A_1 + \lambda_1, A_2 + \lambda_2 \text{Id}, \dots, A_k + \lambda_k)$ (resp. $(B_1 + \mu_1, B_2 + \mu_2, \dots, B_k + \mu_k)$) and $\mu(z_0), \mu_{\mathbb{C}}(z_0)$ are block diagonal. Hence we search matrices $A_{i,j}, B_{i,j} \in \mathcal{M}_{n_i, n_j}(\mathbb{C})$, $i < j$ such that for all $i < j$:

$$(4.9) \quad \begin{cases} (A_i^* + \bar{\lambda}_i) A_{i,j} - A_{i,j} (A_j^* + \bar{\lambda}_j) + (B_i^* + \bar{\mu}_i) B_{i,j} - B_{i,j} (B_j^* + \bar{\mu}_j) \\ \quad + Q_1(i, j) + Q_2(i, j)_2 = x_i x_j^* \\ - (B_i + \mu_i) A_{i,j} + A_{i,j} (B_j + \mu_j) + (A_i + \lambda_i) B_{i,j} - B_{i,j} (A_j + \lambda_j) + Q_3(i, j) = 0 \end{cases}$$

where $Q_1(i, j)$ (resp. $Q_2(i, j)$) is a quadratic expression depending on the $A_{\alpha, \beta}$'s (resp. in the $B_{\alpha, \beta}$) and $Q_3(i, j)$ is bilinear in $A_{\alpha, \beta}$'s and $B_{\alpha, \beta}$. For $\tau > 0$ small enough, with the same arguments given in the preceding paragraph, the implicit function theorem implies

Lemma 4.1. *The equations (4.9) has a solution $A_{i,j}, B_{i,j} \in \mathcal{M}_{n_i, n_j}(\mathbb{C})$, $i < j$ which depends smoothly on $\zeta \in \mathcal{O}_{\mathbf{p}}$, moreover we have that*

$$\sum_{i < j} \|A_{i,j}\|^2 + \|B_{i,j}\|^2 = O\left(\frac{1}{|\mathbf{q}|^2}\right).$$

And the derivative of the map $\zeta \mapsto (A_{i,j}, B_{i,j})$ is bounded by $O\left(\frac{1}{|\mathbf{q}|^2}\right)$.

Then we obtain $z_0(\zeta) = (A^0, B^0, x^0, 0) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \times \mathbb{C}^n \times (\mathbb{C}^n)^*$ an almost solution of the moment map equation :

$$\begin{cases} [A^0, B^0] = 0 \\ 2i\mu(z_0) - t = O\left(\frac{1}{|\mathbf{q}|^2}\right) \end{cases}$$

More precisely, the off block diagonal terms of the moment map equations are zero. We will now use an argument that we learned in a paper of S. Donaldson [8][Proposition 17] : we will find $h = ik$ a Hermitian matrix such that if

$$z_h = e^{ik} \cdot z_0 = (e^h A^0 e^{-h}, e^h B^0 e^{-h}, e^h \cdot x^0, 0)$$

then $2i\mu(z_h) - t \text{Id} = 0$ and $\mu_{\mathbb{C}}(z_h) = 0$ (this latter condition being obvious).

By the induction hypothesis *ii*) and if τ is small enough and R large enough then we have

$$\forall \eta \in \mathfrak{u}_n, \|l_{z_0}(\eta)\| \geq C|\eta|$$

the constant C being uniform on $\mathcal{O}_{\mathfrak{p}}$. Hence if $h = i\eta$ with

$$\|h\| \leq \delta := \min \left\{ 1, \frac{Ce^{-2}}{4|z_0|} \right\}$$

then

$$\forall \eta \in \mathfrak{u}_n, \|l_{z_h}(\eta)\| \geq \frac{C}{2}|\eta|$$

So as soon as we have $|\mu(z_0) - \frac{t}{2i} \text{Id}| < \frac{4}{C^2} \delta$, the proposition 17 in [8] furnishes a $h = ik$ with $\mu(e^h \cdot z_0) = \frac{t}{2i} \text{Id}$ with

$$\|h\| \leq \frac{4}{C^2} \left| \mu(z_0) - \frac{t}{2i} \text{Id} \right|.$$

But when R is large enough, the condition $|\mu(z_0) - \frac{t}{2i} \text{Id}| < \frac{4}{C^2} \delta$ is satisfied, hence there is $h = ik$ a Hermitian matrix such that $2i\mu(z_h) - t \text{Id} = 0$ and $\mu_{\mathbb{C}}(z_h) = 0$.

We need to recall how h is found. For $z \in \mathbb{M}$ we have a linear map $l_z : \mathfrak{u}_n \rightarrow T_z \mathbb{M} \simeq \mathbb{M}$ and l_z^* its adjoint; by definition of the moment map we have $l_z^* = d\mu(z) \circ I$. The endomorphism Q_z of \mathfrak{u}_n is given by $Q_z = l_z^* l_z$. Then for every $h = ik$, with $|k| < \delta$, Q_{z_h} is invertible and $Q_{z_h}^{-1}$ has a operator norm bounded by $4C^{-2}$. Let $a(z) = Q_z^{-1} (\mu(z) - \frac{t}{2i} \text{Id})$, we follow the maximal solution of the equation

$$(4.10) \quad \frac{dz}{ds} = -il_z(a(z)),$$

starting from z_0 at $s = 0$. By definition we have

$$\frac{d\mu(z_s)}{ds} = - \left(\mu(z_s) - \frac{t}{2i} \text{Id} \right)$$

hence

$$\mu(z_s) - \frac{t}{2i} \text{Id} = e^{-s} \left(\mu(z_0) - \frac{t}{2i} \text{Id} \right);$$

in fact $z_s = g_s \cdot z_0$ where

$$\frac{dg_s}{ds} = ia(z_s) \cdot g_s, \quad g_s \in GL_n(\mathbb{C})$$

The arguments of [8] insures that the maximum solution of (4.10) is defined on $[0, +\infty[$. and if $g_s = e^{\eta_s} e^{h_s}$ where $\eta_s \in \mathfrak{u}_n$ and h_s is Hermitian, then $|h_s| \leq \delta$. So that we also get :

$$\|\dot{g}_s\| \leq \frac{4}{C^2} \left| \mu(z_0) - \frac{t}{2i} \text{Id} \right| e^{-s} e^{\delta}$$

hence $g_{\infty} = \lim_{s \rightarrow +\infty} g_s$ exists and

$$(4.11) \quad \|g_{\infty} - \text{Id}\| \leq \frac{4e^{\delta}}{C^2} \left| \mu(z_0) - \frac{t}{2i} \text{Id} \right| = O \left(\frac{1}{|\mathbf{q}|^2} \right).$$

We clearly have $2i\mu(g_\infty.z_0) = t \text{Id}$ and $h = ik$ is given by $e^{2h} = g_\infty^* g_\infty$ i.e. the polar decomposition of g_∞ is $g_\infty = e^{\eta_\infty} e^h$. Moreover if $s \geq 0$, then the operator norm of $l_{z_s} Q_{z_s}^{-1}$ remains less than $2/C$ hence

$$(4.12) \quad \|g_\infty.z_0 - z_0\| \leq \frac{2}{C} \left| \mu(z_0) - \frac{t}{2i} \text{Id} \right| = O\left(\frac{1}{|\mathbf{q}|^2}\right).$$

The implicit function theorem told us that h depends smoothly on z_0 hence on $\zeta \in \mathcal{O}_p$, indeed

$$\frac{d}{dt} \Big|_{t=0} \mu(e^{tik}.z) = Q_z(k).$$

This map will be called : $\zeta \in \mathcal{O}_p \rightarrow h(\zeta) \in i\mathfrak{u}_n$. The following lemma gives an estimate of the size of the derivative of h

Lemma 4.2. *Let $\mathbf{v} \in T_\zeta \mathcal{O}_p$ be a vector of unit length then*

$$\|dh(\zeta).\mathbf{v}\| = O\left(\frac{1}{|\mathbf{q}|^2}\right).$$

Proof. Let $\dot{z}_0 = dz_0(\zeta_0).\mathbf{v}$ and $\dot{h} = dh(\zeta).\mathbf{v}$, we also let $v \in \mathbb{M}$ be the vector $v := (\delta A, \delta B, \delta x, 0)$ where if

$$\mathbf{v} = (((\delta\lambda_1, \delta\mu_1), \dots, (\delta\lambda_k, \delta\mu_k)), (\delta A_1, \dots, \delta A_k), (\delta B_1, \dots, \delta B_k), (\delta x_1, \dots, \delta x_k))$$

then δA (resp. δB) is a block diagonal matrix with diagonal $(\delta A_1 + \delta\lambda_1 \text{Id}_{n_1}, \dots, \delta A_k + \delta\lambda_k \text{Id}_{n_k})$ (resp. $(\delta B_1 + \delta\mu_1 \text{Id}_{n_1}, \dots, \delta B_k + \delta\mu_k \text{Id}_{n_k})$) and $\delta x = ((\delta x_1, \dots, \delta x_k))$.

We have

$$d\mu(z_h). \left(D \exp(h) \dot{h}.z_0 + e^h \dot{z}_0 \right) = 0$$

Recall that :

$$D \exp(h) \dot{h} = \frac{e^{\text{ad } h} - \text{Id}}{\text{ad } h} \dot{h} e^h.$$

Let $i\dot{\eta}$ be the Hermitian part of $D \exp(h) \dot{h}$ and $\dot{\xi}$ be its skew Hermitian part. Then

$$d\mu(z_h). (D \exp(h) \dot{h}.z_0) = d\mu(z_h)(i l_{z_h} \dot{\eta}) + d\mu(z_h)(l_{z_h} \dot{\xi}) = Q_{z_h}(\dot{\eta}).$$

Moreover from the construction of z_0 and the lemma (4.1), we obtain easily that

$$d\mu(z_0)(\dot{z}_0) = O\left(\frac{1}{|\mathbf{q}|^2}\right);$$

and

$$\dot{z}_0 = v + O\left(\frac{1}{|\mathbf{q}|^2}\right) |v|.$$

So if $k \in U(n)$ is such that $g_\infty = k e^h$ then

$$\begin{aligned} \text{Ad}(k) d\mu(z_h). (e^h \dot{z}_0) &= d\mu(g_\infty.z_0). (g_\infty \dot{z}_0) \\ &= d\mu(g_\infty.z_0). ((g_\infty - \text{Id}).\dot{z}_0) + d\mu(g_\infty.z_0 - z_0).\dot{z}_0 + d\mu(z_0)(\dot{z}_0) \end{aligned}$$

Hence

$$Q_{z_h}(\dot{\eta}) + d\mu(z_h). (k^{-1}.(g_\infty - \text{Id}).\dot{z}_0) = O\left(\frac{1}{|\mathbf{q}|^2}\right).$$

We now make the scalar product of this quantity with $\dot{\eta}$ and we obtain :

$$\begin{aligned} \|l_{z_h}(\dot{\eta})\|^2 &\leq O\left(\frac{1}{|\mathbf{q}|^2}\right) \|\dot{\eta}\| - \langle l_{z_h}(\dot{\eta}), k^{-1} \cdot I (g_\infty - \text{Id}) \cdot \dot{z}_0 \rangle \\ &\leq O\left(\frac{1}{|\mathbf{q}|^2}\right) (\|\dot{\eta}\| + \|l_{z_h}(\dot{\eta})\|) \end{aligned}$$

But our construction gives that

$$\|\dot{\eta}\| \leq \frac{2}{C} \|l_{z_h}(\dot{\eta})\|,$$

hence we obtain :

$$\|\dot{\eta}\| \leq \frac{2}{C} \|l_{z_h}(\dot{\eta})\| = O\left(\frac{1}{|\mathbf{q}|^2}\right).$$

Now \dot{h} is a Hermitian matrix and $\|h\| = O(|\mathbf{q}|^{-2})$ hence by definition of $\dot{\eta}$ and $\dot{\xi}$, we have

$$\dot{h} - i\dot{\eta} = O(|\mathbf{q}|^{-2}).$$

Hence the lemma. \square

We note that it is straightforward to verify the point *ii)* at z_h because by construction

$$\forall \eta \in \mathfrak{u}_n, |l_{z_h}(\eta)| \geq \frac{C}{2} \|\eta\|.$$

We have build a map $f_{\mathfrak{p}}$ from $\mathcal{O}_{\mathfrak{p}}$ to \mathbb{L}_t whose value at a point $\zeta = (\mathbf{q}, \mathbf{A}, \mathbf{B}, \mathbf{x}) \in \mathcal{O}_{\mathfrak{p}}$ is the z_h constructed before. This map is $U(n_1) \times U(n_2) \times \dots \times U(n_k)$ -equivariant hence it induces a map

$$\psi_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}} / (U(n_1) \times U(n_2) \times \dots \times U(n_k)) \rightarrow \text{Hilb}_0^n(\mathbb{C}^2)$$

We remark that adjusting ϵ, R, τ , we have

$$\mathcal{C}_{\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{p}} / (U(n_1) \times U(n_2) \times \dots \times U(n_k)) \subset \text{Hilb}_0^p(\mathbb{C}^2) \times (\mathbb{C}^2)_0^k.$$

Where the last inclusion is an isometry if $\text{Hilb}_0^p(\mathbb{C}^2) \times (\mathbb{C}^2)_0^k$ is endowed with the product metric. We now want to compare the metric $\psi_{\mathfrak{p}}^* g_N$ and the product metric on $\text{Hilb}_0^p(\mathbb{C}^2) \times (\mathbb{C}^2)_0^k$. Let \mathbf{v} be a vector of $T_{\zeta} \mathcal{O}_{\mathfrak{p}}$ which is orthogonal to the $U(n_1) \times U(n_2) \times \dots \times U(n_k)$ orbit of ζ . As before, we defined $f(\zeta) = z_h = e^h \cdot z_0, \dot{h}, v..$

Recall that we have denote by Π_q the orthogonal projection onto $\text{Im } l_q$. Hence we need to compare

$$\|(\text{Id} - \Pi_{z_h}) \cdot df_{\mathfrak{p}}(\zeta) \cdot \mathbf{v}\|^2 = \|df_{\mathfrak{p}}(\zeta) \cdot \mathbf{v}\|^2 - \|\Pi_{z_h} \cdot df_{\mathfrak{p}}(\zeta) \cdot \mathbf{v}\|^2 \text{ and } \|\mathbf{v}\|^2.$$

But

$$df_{\mathfrak{p}}(\zeta) \cdot \mathbf{v} = l_{z_h}(\dot{\xi}) + i l_{z_h}(\dot{\eta}) + e^h \cdot \dot{z}_0$$

Hence

$$(\text{Id} - \Pi_{z_h}) \cdot df_{\mathfrak{p}}(\zeta) \cdot \mathbf{v} = (\text{Id} - \Pi_{z_h}) \cdot (i l_{z_h}(\dot{\eta}) + e^h \cdot \dot{z}_0).$$

But we have already seen that

$$\|l_{z_h}(\dot{\eta})\|^2 = O\left(\frac{1}{|\mathbf{q}|^4}\right).$$

but $il_{z_h}(\dot{\eta})$ is orthogonal to $T\mathbb{L}_t$ hence to the range of Π_{z_h} , we also have

$$\begin{aligned} \langle (\text{Id} - \Pi_{z_h})(il_{z_h}(\dot{\eta})), (\text{Id} - \Pi_{z_h}).(e^h \cdot \dot{z}_0) \rangle &= \langle (\text{Id} - \Pi_{z_h}).il_{z_h}(\dot{\eta}), (e^h \cdot \dot{z}_0) \rangle \\ &= \langle il_{z_h}(\dot{\eta}), (e^h \cdot \dot{z}_0) \rangle \\ &= -\langle \dot{\eta}, d\mu(z_h)(e^h \cdot \dot{z}_0) \rangle \end{aligned}$$

But

$$d\mu(z_h)((df_{\mathfrak{p}}(\zeta) \cdot \mathbf{v}) = 0 = d\mu(z_h)(il_{z_h}(\dot{\eta}) + e^h \cdot \dot{z}_0)$$

so that

$$\begin{aligned} \langle (\text{Id} - \Pi_{z_h})(il_{z_h}(\dot{\eta})), (\text{Id} - \Pi_{z_h}).(e^h \cdot \dot{z}_0) \rangle &= -\langle \dot{\eta}, d\mu(z_h)(e^h \cdot \dot{z}_0) \rangle \\ &= \langle \dot{\eta}, d\mu(z_h)(il_{z_h}(\dot{\eta})) \rangle = \|il_{z_h}(\dot{\eta})\| \\ &= O\left(\frac{1}{|\mathbf{q}|^4}\right). \end{aligned}$$

It remains now to estimate

$$\|(\text{Id} - \Pi_{z_h})(e^h \cdot \dot{z}_0)\|^2 = \|e^h \cdot \dot{z}_0\|^2 - \|\Pi_{z_h}(e^h \cdot \dot{z}_0)\|^2.$$

But $\Pi_{z_h} = l_{z_h} Q_{z_h}^{-1} l_{z_h}^*$, and

$$l_{z_h}^*(e^h \cdot \dot{z}_0) = d\mu(z_h)(ie^h \cdot \dot{z}_0).$$

We have already noticed that

$$I \cdot \dot{z}_0 = dz_0(\zeta) \cdot (I \cdot \mathbf{v}) + w$$

where $w = O(|\mathbf{q}|^{-2})$. So

$$l_{z_h}^*(e^h \cdot \dot{z}_0) = d\mu(z_h)(e^h dz_0(\zeta) \cdot (I \cdot \mathbf{v})) + l_{z_h}^*(w)$$

And the proof of the lemma (4.2), furnishes a $w' = O(|\mathbf{q}|^{-2})$ such that

$$d\mu(z_h)(e^h dz_0(\zeta) \cdot (I \cdot \mathbf{v})) = d\mu(z_h)(w') + O(|\mathbf{q}|^{-2})$$

as the operator norm of $l_{z_h} Q_{z_h}^{-1}$ is bounded by $2/C$ we have obtained :

$$\|\Pi_{z_h}(e^h \cdot \dot{z}_0)\|^2 = O\left(\frac{1}{|\mathbf{q}|^4}\right).$$

Hence we have obtain :

$$\begin{aligned} \psi_{\mathfrak{p}}^* g_N(\mathbf{v}, \mathbf{v}) &= \|e^h \cdot \dot{z}_0\|^2 + O\left(\frac{1}{|\mathbf{q}|^4}\right) \|v\|^2 \\ &= \|\dot{z}_0\|^2 + 2\langle \dot{z}_0, h \dot{z}_0 \rangle + O\left(\frac{1}{|\mathbf{q}|^4}\right) \|v\|^2. \end{aligned}$$

By construction

$$|\dot{z}_0|^2 = \|v\|^2 + O\left(\frac{1}{|\mathbf{q}|^4}\right) \|v\|^2$$

And if $\mathbf{v} = (\delta q, \delta A_1, \delta A_2, \dots, \delta A_k, \delta B_1, \delta B_2, \dots, \delta B_k, 0)$ and if $h_{i,j}$ are the block of h of size $n_i \times n_j$ then

$$\begin{aligned} \langle \dot{z}_0, h\dot{z}_0 \rangle &= \langle v, hv \rangle + O\left(\frac{1}{|\mathbf{q}|^4}\right) \|v\|^2 \\ &= \sum_j \langle \delta A_j, [h_{j,j}, \delta A_j] \rangle + \langle \delta B_j, [h_{j,j}, \delta B_j] \rangle + O\left(\frac{1}{|\mathbf{q}|^4}\right) \|v\|^2 \\ &= \sum_j \langle [\delta A_j, \delta A_j^*] + [\delta B_j, \delta B_j^*], h_{j,j} \rangle + O\left(\frac{1}{|\mathbf{q}|^4}\right) \|v\|^2 \\ &= O\left(\frac{1}{\sigma^2 |\mathbf{q}|^2}\right) \|v\|^2 \end{aligned}$$

according to the hypothesis *iii*).

In order to finish the proof we need to check the property *iii*) at the point z_h . With what has been proved in the preceding paragraph, we only need to check that if $(\delta A, \delta B, 0, 0)$ is a unitary vector in the tangent space of \mathbb{L}_t at z_h and orthogonal to $U(n)$ orbit of z_h then $[\delta A_j, \delta A_j^*] + [\delta B_j, \delta B_j^*]$ is bounded. This is evident.

5. CONCLUSION

With the previous asymptotic of Nakajima's metric, we'll show that Nakajima's metric coincides with Joyce's one ; a way for proving such a result would be to verify the estimate 3.1) for the orders $l \geq 1$; this is probably possible with some extra work, however we'll give here a different proof which follows the classical proof of the unicity for the solution of the Calabi-Yau problem. Moreover our argument gives a new analytical result on mapping property of the Laplace operator on QALE space. For new results which extended Joyce's ones and which go further than our result, there is a forthcoming work of A. Degeratu and R. Mazzeo [7].

We have already seen that Kronheimer's classification of hyperkähler ALE 4–dimensional manifold implies that on $\text{Hilb}_0^2(\mathbb{C}^2) \simeq T^*\mathbb{P}^1(\mathbb{C})$, Nakajima's metric is the Eguchi-Hansen metric. We are going to prove our result by induction on n . Hence we now assume that up to a scaled factor, Joyce's and Nakajima's metrics coincide on $\text{Hilb}_0^l(\mathbb{C}^2)$ for all integer $l < n$. We consider g the Joyce's metric on $\text{Hilb}_0^n(\mathbb{C}^2)$ and ω the Kähler form associated to g (for the complex structure I) and for simplicity of forthcoming notation, we denote by g' Nakajima's metric on $\text{Hilb}_0^n(\mathbb{C}^2)$ with associated Kähler form ω' .

5.1. Comparison of the two metrics. The second group of cohomology of $\text{Hilb}_0^n(\mathbb{C}^2)$ has dimension 1 and a cycle dual to a basis of $H^2(\text{Hilb}_0^n(\mathbb{C}^2), \mathbb{R})$ is given by the image of a holomorphic map $f_n : \mathbb{P}^1(\mathbb{C}) \rightarrow \text{Hilb}_0^n(\mathbb{C}^2)$ such that if $\pi : \text{Hilb}_0^n(\mathbb{C}^2) \rightarrow (\mathbb{C}^2)_0^n / S_n$ then the image of f_n is $\mathbb{P}^1(\mathbb{C}) \simeq \pi^{-1}\{((0, 0), q')\}S_n$ for $q' \in (\mathbb{C}^2 \setminus \{0\})_0^{n-2}$. We can assume that

$$\int_{\mathbb{P}^1(\mathbb{C})} f_n^* \omega = \int_{\mathbb{P}^1(\mathbb{C})} f_n^* \omega'$$

Moreover, for each partition \mathbf{p} of $\{1, 2, \dots, n\}$, we have on $\mathcal{C}_{\mathbf{p}} \subset \text{Hilb}_0^{\mathbf{p}}(\mathbb{C}^2) \times V_{\mathbf{p}}$

$$\psi_{\mathbf{p}}^* g = g_{\mathbf{p}} + \text{eucl} + O\left(\frac{1}{\sigma^2 \rho^2}\right)$$

and

$$\psi_{\mathfrak{p}}^* g' = g_{\mathfrak{p}}' + \text{eucl} + O\left(\frac{1}{\sigma^2 \rho^2}\right)$$

where $g_{\mathfrak{p}}$ (resp. $g'_{\mathfrak{p}}$) is the sum of the Joyce's (resp. Nakajima's) metric on $\text{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) \simeq \text{Hilb}_0^{n_1}(\mathbb{C}^2) \times \text{Hilb}_0^{n_2}(\mathbb{C}^2) \times \dots \times \text{Hilb}_0^{n_k}(\mathbb{C}^2)$. However $f_n(\mathbb{P}^1(\mathbb{C}))$ is homologous to $C_{\mathfrak{p},j,v} := \psi_{\mathfrak{p}}\{(y_1, \dots, y_{n_j-1}) \times f_{n_j}(\mathbb{P}^1(\mathbb{C})) \times \{y_{n_j+1}, \dots, y_{n_k}\} \times \{v\}$ where $y_j \in \text{Hilb}_0^{n_j}(\mathbb{C}^2)$ and $v \in V_{\mathfrak{p}}$, let g_j (resp. g'_j) be the Joyce's (resp. Nakajima's) metric on $\text{Hilb}_0^{n_j}(\mathbb{C}^2)$ and ω_j (resp. ω'_j) its Kähler form ; that is to say $g_{\mathfrak{p}} = g_1 + g_2 + \dots + g_k$ and $g'_{\mathfrak{p}} = g'_1 + g'_2 + \dots + g'_k$. We have

$$\begin{aligned} \int_{C_{\mathfrak{p},j,v}} \omega &= \int_{f_{n_j}(\mathbb{P}^1(\mathbb{C}))} \omega_j + O\left(\frac{1}{\sigma^2 \rho^2}\right) \\ &= \int_{C_{\mathfrak{p},j,v}} \omega' \\ &= \int_{f_{n_j}(\mathbb{P}^1(\mathbb{C}))} \omega'_j + O\left(\frac{1}{\sigma^2 \rho^2}\right) \end{aligned}$$

In particular letting $\|v\|$ going to ∞ , we obtain

$$\int_{f_{n_j}(\mathbb{P}^1(\mathbb{C}))} \omega'_j = \int_{f_{n_j}(\mathbb{P}^1(\mathbb{C}))} \omega_j$$

Our induction hypothesis yields that $g_j = g'_j$ for all j , and eventually, we have proved that

$$g - g' = O\left(\frac{1}{\sigma^2 \rho^2}\right)$$

5.2. Coincidence of Joyce's and Nakajima's metrics. Following the classical proof of the unicity of the solution of the Calabi-Yau problem, we would like to find a good function ϕ such that

$$\omega - \omega' = i\partial\bar{\partial}\phi.$$

However it is not easy because the weight $\sigma^{-2}\rho^{-2}$ is critical in Joyce's analysis on QALE manifold. To circumvent this difficulty, we remark that both metrics g and g' have a \mathbb{S}^1 invariance property coming from the diagonal action of \mathbb{S}^1 on $(\mathbb{C}^2)_0^n/S_n$. For Joyce's metric it comes from the unicity result of the QALE Kähler Einstein metric asymptotic to $(\mathbb{C}^2)_0^n/S_n$. For Nakajima's metric, the action of \mathbb{S}^1 on \mathbb{M} is the following : if $e^{i\theta} \in \mathbb{S}^1$ and if $z = (A, B, x, 0) \in \mathbb{L}_t$ then $e^{i\theta} \cdot z := (e^{i\theta} A, e^{i\theta} B, e^{i\theta} x, 0) \in \mathbb{L}_t$. And this action is isometric. This \mathbb{S}^1 action is holomorphic for the complex structure I but not for the complex structures J and K . Let X be the g or g' Killing field associated to the infinitesimal action of $\eta = i/2$. Then X has linear growth on $\text{Hilb}_0^n(\mathbb{C}^2)$ that is to say there is a constant c such that

$$X(z) \leq c(\rho(z) + 1).$$

Moreover if ω_1 is the Kähler form of (g, J) , ω_2 is the Kähler form of (g, K) and ω'_1 and ω'_2 are the corresponding form associated to the metric g' then

$$\omega_1 = d(i_X \omega_2) \text{ and } \omega'_1 = d(i_X \omega'_2)$$

Hence if we let

$$\beta = i_X \omega_2 - i_X \omega'_2$$

then we have

$$\omega_1 - \omega'_1 = d\beta \text{ and } \beta = O\left(\frac{1}{\sigma^2 \rho}\right)$$

We work now in the Kähler manifold $(\text{Hilb}_0^n(\mathbb{C}^2), g, J)$, the following analytical result is the key point of our proof :

Proposition 5.1. *There is a $(0, 1)$ -form α on $\text{Hilb}_0^n(\mathbb{C}^2)$ such that*

$$\alpha = O\left(\frac{\log(\rho + 2)}{\rho}\right)$$

and

$$\beta^{0,1} = \Delta_{\bar{\partial}}\alpha = \bar{\partial}\bar{\partial}^*\alpha + \bar{\partial}^*\bar{\partial}\alpha.$$

We first explain how we can prove that $\omega_1 = \omega'_1$ with this proposition. This proposition will be proved in the next subsection.

The 1-form $\Phi = \bar{\partial}^*\bar{\partial}\alpha$ satisfies $\bar{\partial}\beta^{0,1} = 0 = \bar{\partial}\Phi$ and $\bar{\partial}^*\Phi = 0$. Moreover the metric g has by definition bounded geometry, hence we have the following uniform in $x \in \text{Hilb}_0^n(\mathbb{C}^2)$ local elliptic estimate :

$$\begin{aligned} \|\Phi\|_{L^2(B(x,1))} &= \|\bar{\partial}^*\bar{\partial}\alpha\|_{L^2(B(x,1))} \leq c\|\Delta_{\bar{\partial}}\alpha\|_{L^2(B(x,2))} + c'\|\alpha\|_{L^2(B(x,2))} \\ &\leq O\left(\frac{\log(\rho + 2)}{\rho}\right) \end{aligned}$$

But Φ being harmonic we also have a uniform estimate

$$|\Phi(x)| \leq c\|\Phi\|_{L^2(B(x,1))}.$$

Hence we obtain that

$$\Phi = O\left(\frac{\log(\rho + 2)}{\rho}\right).$$

But the Ricci curvature of g is zero hence the Bochner formula and the Kato inequality implies that $|\Phi|$ is a subharmonic function hence Φ is zero by the maximum principle. And we get $\beta^{0,1} = \bar{\partial}\bar{\partial}^*\alpha$, the same argument shows that we can find a $(1, 0)$ -form $\tilde{\alpha}$ such that $\beta^{1,0} = \partial\partial^*\tilde{\alpha}$. Hence if we let

$$i\phi = \bar{\partial}^*\alpha - \partial^*\tilde{\alpha}$$

then we have

$$d\beta = i\partial\bar{\partial}\phi.$$

Again the same argument as before using the fact that g has bounded geometry, implies that

$$\phi = O\left(\frac{\log(\rho + 2)}{\rho}\right).$$

Both ω_1 and ω'_1 are Kähler Einstein with zero scalar curvature hence there is a pluriharmonic function f such that

$$\omega_1^m = e^f (\omega'_1)^m$$

But we also have

$$f = O\left(\frac{1}{\sigma^2 \rho^2}\right).$$

By the maximum principle we deduce that $f = 0$. We finish the proof with a classical argument : the function ϕ is subharmonic for the metric g [4][exposé VI, lemma 1.6] and decay at infinity hence by the maximum principle ϕ is negative ; but reversing the role of g and g' , $-\phi$ is also subharmonic for the metric g' and $-\phi$ is positive and decay at infinity hence ϕ is zero.

5.3. Proof of the analytical result. We first remark that because $(\text{Hilb}_0^n(\mathbb{C}^2), g)$ is asymptotic to the Euclidean metric on $(\mathbb{C}^2)_0^n / S_n$, we have

$$\lim_{r \rightarrow \infty} \frac{\text{vol } B(x, r)}{r^d} = \frac{w_d}{n!}$$

where $d = 4(n - 1)$ is the real dimension of $\text{Hilb}_0^n(\mathbb{C}^2)$ and w_d is the volume of the unit ball in \mathbb{R}^d . The Bishop-Gromov inequality tells us that for any point $x \in \text{Hilb}_0^n(\mathbb{C}^2)$:

$$\frac{w_d r^d}{n!} \leq \text{vol } B(x, r) \leq w_d r^d.$$

The result of P. Li and S-T. Yau implies that the Green kernel G of the metric g (that is to say the Schwartz kernel of the operator Δ^{-1}) satisfies [20] :

$$G(x, y) \leq \frac{c}{d(x, y)^{d-2}}.$$

Moreover because g is Ricci flat, the Hodge-deRham operator acting on 1 forms is the rough Laplacian :

$$\forall v \in C_0^\infty(T^* \text{Hilb}_0^n(\mathbb{C}^2)), \quad \Delta = dd^* + d^* d = \nabla^* \nabla$$

Hence the Kato inequality implies that if $\vec{G}(x, y)$ is the Schwartz kernel of the operator Δ^{-1} , then it satisfies

$$|\vec{G}(x, y)| \leq G(x, y) \leq \frac{c}{d(x, y)^{d-2}}.$$

The proposition (5.1) will be a consequence of the following lemma

Lemma 5.2. *If $f \in L_{\text{loc}}^\infty(\text{Hilb}_0^n(\mathbb{C}^2))$ is a non negative function which satisfies*

$$f = O\left(\frac{1}{\sigma^2 \rho}\right),$$

then

$$u(x) = \int_{\text{Hilb}_0^n(\mathbb{C}^2)} \frac{f(y)}{d(x, y)^{d-2}} dy$$

is well defined and satisfies

$$u = O\left(\frac{\log(\rho + 2)}{\rho}\right).$$

Proof. Let $o \in \text{Hilb}_0^n(\mathbb{C}^2)$ be a fixed point. And we can assume that $\rho(x) = d(o, x)$. We remark that u is well defined indeed there is a constant c such that for $R > 1$ then

$$\int_{B(o, R)} f \leq cR^{d-3}.$$

As a matter of fact, the function $\frac{1}{\sigma^2 \rho}$ is asymptotic to a homogeneous function h of degree -3 on $(\mathbb{C}^2)_0^n / S_n$ $h(r\theta) = r^{-3} \bar{h}(\theta)$ where \bar{h} is a positive function on \mathbb{S}^{2d-1} / S_n ; this function \bar{h} is singular on the singular locus of \mathbb{S}^{2d-1} / S_n ; we call Σ this singular locus. but \bar{h} behaves like $d(., \Sigma)^{-2}$ near Σ but the real co dimension of Σ is 4 hence \bar{h} is integrable on \mathbb{S}^{2d-1} / S_n and we have

$$\lim_{R \rightarrow \infty} R^{3-d} \int_{B(o, R)} \frac{1}{\rho \sigma^2} = \frac{1}{d-3} \int_{\mathbb{S}^{2d-1} / S_n} \bar{h}.$$

In order to finish our estimate, we must find a constant c such that if $\rho(x) \geq 10$ then

$$F(x) = \int_{\text{Hilb}_0^n(\mathbb{C}^2)} \frac{1}{d(x, y)^{d-2}} \frac{1}{\rho(y) \sigma(y)^2} dy \leq c \frac{\log \rho(x)}{\rho(x)}$$

We decompose

(5.1)

$$\text{Hilb}_0^n(\mathbb{C}^2) = (B(o, 2\rho(x)) \setminus B(x, \rho(x)/2)) \cup B(x, \rho(x)/2) \cup (\text{Hilb}_0^n(\mathbb{C}^2) \setminus B(o, 2\rho(x))),$$

then we have $F = F_1 + F_2 + F_3$ where F_i is the integral of $d(x, y)^{2-d} \rho^{-1} \sigma^{-2}$ on the i^{th} region of the decomposition (5.1). The first and the last integrals are easy to estimate :

$$F_1(x) \leq \left(\frac{2}{\rho(x)} \right)^{d-2} \int_{B(o, 2\rho(x))} \frac{1}{\rho(x)\sigma(x)^2} \leq C \frac{1}{\rho(x)}.$$

Concerning F_3 we have

$$\begin{aligned} F_3(x) &= \int_{\text{Hilb}_0^n(\mathbb{C}^2) \setminus B(o, 2\rho(x))} \frac{2^{d-2}}{d(x, y)^{d-2}} \frac{1}{\rho(y)\sigma(y)^2} dy \\ &\leq \int_{\text{Hilb}_0^n(\mathbb{C}^2) \setminus B(o, 2\rho(x))} \frac{2^{d-2}}{\rho(y)^{d-1}} \frac{1}{\sigma(y)^2} dy \\ &\leq \sum_{k=1}^{\infty} \int_{B(o, 2^{k+1}\rho(x)) \setminus B(o, 2^k\rho(x))} \frac{2^{d-2}}{\rho(y)^{d-1}} \frac{1}{\sigma(y)^2} dy \\ &\leq \sum_{k=1}^{\infty} \frac{1}{(2^k\rho(x))^{d-2}} \int_{B(o, 2^{k+1}\rho(x))} \frac{1}{\rho(y)\sigma(y)^2} dy \\ &\leq C \sum_{k=1}^{\infty} \frac{1}{(2^k\rho(x))^{d-2}} (2^{k+1}\rho(x))^{d-3} \\ &\leq C' \frac{1}{\rho(x)}. \end{aligned}$$

It remains to estimate F_2 : We have

$$F_2(x) \leq \frac{2}{\rho(x)} \int_{B(x, \rho(x)/2)} \frac{1}{d(x, y)^{d-2}} \frac{1}{\sigma(y)^2} dy.$$

Let $V(\tau) = \int_{B(x, \tau)} \frac{1}{\sigma(y)^2} dy$ and note dV the Riemann-Stieljes measure associated to the increasing function V . We have

$$\begin{aligned} (5.2) \quad \int_{B(x, \rho(x)/2)} \frac{1}{d(x, y)^{d-2}} \frac{1}{\sigma(y)^2} dy &= \int_0^{\rho(x)/2} \frac{1}{\tau^{d-2}} dV(\tau) \\ &= \frac{V(\rho(x)/2)}{(\rho(x)/2)^{d-2}} + (d-2) \int_0^{\rho(x)/2} \frac{V(\tau)}{\tau^{d-1}} d\tau. \end{aligned}$$

We will estimate V : if we note S the pull back to $\text{Hilb}_0^n(\mathbb{C}^2)$ of the singular locus of $(\mathbb{C}^2)_0^n / S_n$ and $\mathcal{O} = \{y \in \text{Hilb}_0^n(\mathbb{C}^2), \text{ such that } \sigma(y) \geq 2\}$ then we have $V(\tau) = V_1(\tau) + V_2(\tau)$ where V_1 is the integral over $B(x, \tau) \cap \mathcal{O}$ and V_2 is the integral over $B(x, \tau) \setminus \mathcal{O}$.

V_1 is easy to estimate because on this region, σ^{-2} is bounded hence

$$(5.3) \quad V_1(\tau) \leq C \text{vol}(B(x, \tau) \cap \mathcal{O}) \leq C \min\{\tau^d, \tau^{d-4}\}.$$

Outside \mathcal{O} the metric is quasi-isometric to the Euclidean metric and we can estimate V_2 by a similar integral on $(\mathbb{C}^2)_0^n / S_n$. Let

$$D = \{q \in (\mathbb{C}^2)_0^n, \text{ such that } \forall i \neq j \quad |q_i - q_j| \geq |q_1 - q_2|\}$$

and $D' = \{q \in D, |q_1 - q_2| \geq 1\}$. D is a fundamental domain for the action of S_n on $(\mathbb{C}^2)_0^n$ and if $\bar{x} \in D$ is such that $S_n \bar{x} = \pi(x)$ then

$$V_2(\tau) \leq \frac{C}{n!} \sum_{\gamma \in S_n} \int_{D' \cap B(\gamma \bar{x}, \tau)} \frac{1}{|q_1 - q_2|^2} dq.$$

We give three different estimates for V_2 according to the relative size of $\sigma(x)$ and τ :

- (1) If $\sigma(x) \leq 3/2$ then for $\tau \in [0, 1/2]$ we have $V_2(\tau) = 0$.
- (2) if $\sigma(x) \geq 3/2$ then for $\tau \leq \sigma(x)/2$ then

$$V_2(\tau) \leq \frac{C}{\sigma(x)^2} \tau^d$$

- (3) and finally if $\tau \geq \sigma(x)/2$ then there is a point $z \in S$ such that $d(x, z) = \sigma(x) - 1$ and if $\bar{z} \in D$ such that $S_n \bar{z} = \pi(z)$ then

$$\int_{D' \cap B(\gamma \bar{x}, \tau)} \frac{1}{|q_1 - q_2|^2} dq \leq \int_{D' \cap B(\gamma \bar{z}, 3\tau)} \frac{1}{|q_1 - q_2|^2} dq \leq C\tau^{d-2}.$$

Now, with the estimate (5.3), it is easy to show that in (5.2) the part coming from V_1 is bounded ; concerning the part coming from V_2 , when $\sigma(x) \leq 3/2$, we get

$$\int_0^{\rho(x)/2} \frac{1}{\tau^{d-2}} dV_2(\tau) \leq C + (d-2) \int_{3/2}^{\rho(x)/2} \frac{C\tau^{d-2}}{\tau^{d-1}} d\tau = C' + C \log \rho(x),$$

and when $\sigma(x) \geq 3/2$, we obtain

$$\begin{aligned} \int_0^{\rho(x)/2} \frac{1}{\tau^{d-2}} dV_2(\tau) &\leq C + (d-2) \int_0^{\sigma(x)/2} \frac{C\tau^d}{\tau^{d-1}\sigma(x)^2} d\tau + (d-2) \int_{\sigma(x)/2}^{\rho(x)/2} \frac{C\tau^{d-2}}{\tau^{d-1}} d\tau \\ &= C' + C \log \left(\frac{\rho(x)}{\sigma(x)} \right) \end{aligned}$$

Hence the result. □

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